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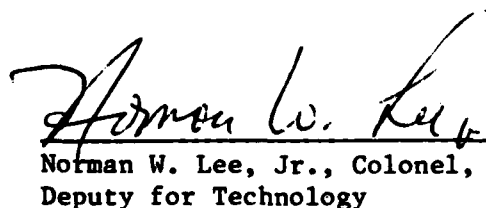
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This technical report has been reviewed and is approved for publication. Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

  
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independent and identically distributed random variables even though the maximum does not behave as such. The asymptotic distribution for the range and midrange are also obtained.

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## 1. INTRODUCTION

The EARMA (1,1) process was introduced by Jacobs and Lewis (Ref. 1). When the parameter  $\rho = 1$  the sequence is given by

$$X_n = \beta \varepsilon_n + U_n A_0 \quad \text{for } n > 1,$$

where  $\varepsilon_n$ 's are i.i.d. with density  $f(x) = \lambda e^{-\lambda x}$ ,  $x > 0$ ,  $\lambda > 0$ ,  $0 < \beta < 1$  and  $\{U_n\}$  is an i.i.d. Bernoulli sequence with  $P[U_n = 1] = 1 - \beta$ .  $A_0$  has density  $f(x)$  and is independent of the  $\varepsilon_n$ 's.

Chernick (Ref. 2) showed that for  $M_n = \max(X_1, X_2, \dots, X_n)$

$$\lim_{n \rightarrow \infty} P[M_n < \frac{\beta(x + \ln(n))}{\lambda}] = \int_0^{\infty} \lambda e^{-\lambda a} \exp(-e^{-x(\beta + (1-\beta)e^{\lambda a \beta})}) da$$

This asymptotic distribution denoted by  $G_\beta(x)$  is not an extreme-value type but is of the general form given by Galambos (Ref. 3) p. 144. The sequence  $\{X_n\}$  is exchangeable.

The conditioning argument given in Chernick (Ref. 2) can be used to obtain the joint asymptotic distribution for the maximum and minimum. This result will be given in the next section. The maximum and minimum are asymptotically independent and the minimum behaves asymptotically similar to the minimum of an i.i.d. sequence of exponential random variables.

Lemma 2.2.1, p. 58, Galambos (Ref. 3) can be applied to show that the range and midrange behave similar to the maximum asymptotically.



## 2. ASYMPTOTIC RESULTS

When  $\beta = 1$  the sequence is i.i.d., and when  $\beta = 0$ ,  $X_n = A_0$  for each  $n$ . So we will not consider these cases.

Theorem 2.1. Let  $\{X_n\}_{n=1}^{\infty}$  be an EARMA (1,1) process with  $\rho = 1$  and  $1 > \beta > 0$ .

Let

$$M_n = \max (X_1, X_2, \dots, X_n) \text{ and}$$

$$W_n = \min (X_1, X_2, \dots, X_n).$$

Then

$$\lim_{n \rightarrow \infty} P[W_n > \frac{y}{\lambda n}, M_n < \beta \frac{(x + \ln(n))}{\lambda}] = e^{-y} G_{\beta}(x).$$

Proof: Let  $K_n$  = number of  $U_i$  which are 0. We consider

$$P[W_n > \frac{y}{\lambda n}, M_n < \beta \frac{(x + \ln(n))}{\lambda}] =$$

$$P[\frac{y}{\lambda n} < X_i < \beta \frac{(x + \ln(n))}{\lambda} \text{ for each } i].$$

We note that  $K_n$  has a binomial distribution with parameters  $n$  and  $1 - \beta$ . Also given  $A_0 = a$ ,  $X_i = \beta \epsilon_i$  if  $U_i = 0$  and  $X_i = \beta \epsilon_i + a$  if  $U_i = 1$ . Let  $u_n = \beta \frac{(x + \ln(n))}{\lambda}$  and  $v_n = \frac{y}{\lambda n}$ . Conditioning on  $K_n$  and  $A_0$  as in Chernick (Ref. 2) we have

$$P[v_n < X_i < u_n \text{ for each } i] =$$

$$\int_0^{\infty} \sum_{k=0}^n \binom{n}{k} \beta^k (1 - \beta)^{n-k} P^k \left[ \frac{v_n}{\beta} < \epsilon_1 < \frac{u_n}{\beta} \right] P^{n-k} \left[ \frac{v_n}{\beta} - \frac{a}{\beta} < \epsilon_1 < \frac{u_n}{\beta} - \frac{a}{\beta} \right] \lambda e^{-\lambda a} da$$

$$\text{For } 0 < a < v_n \quad P\left[\frac{v_n}{\beta} - \frac{a}{\beta} < \varepsilon_1 < \frac{u_n}{\beta} - \frac{a}{\beta}\right] = e^{\lambda a/\beta} e^{-\frac{\lambda v_n}{\beta}} e^{-\frac{\lambda u_n}{\beta}},$$

$$\text{for } v_n < a < u_n \quad P\left[\frac{v_n}{\beta} - \frac{a}{\beta} < \varepsilon_1 < \frac{u_n}{\beta} - \frac{a}{\beta}\right] = 1 - e^{\lambda a/\beta} e^{-\frac{\lambda u_n}{\beta}},$$

$$\text{and for } u_n < a \quad P\left[\frac{v_n}{\beta} - \frac{a}{\beta} < \varepsilon_1 < \frac{u_n}{\beta} - \frac{a}{\beta}\right] = 0.$$

Equation (1) simplifies to

$$\begin{aligned} P[v_n < W_n, M_n < u_n] = & \beta^n \left( e^{-y/(\beta n)} - \frac{e^{-x}}{n} \right)^n \frac{e^{-\beta x}}{\beta} + \int_0^{v_n} \left( \beta \left( e^{-y/(\beta n)} - \frac{e^{-x}}{n} \right) \right. \\ & + (1-\beta) e^{\lambda a/\beta} \left( e^{-y/(\beta n)} - \frac{e^{-x}}{n} \right) \left. \right)^n \lambda e^{-\lambda a} da \\ & + \int_{v_n}^{u_n} \left( \beta \left( e^{-y/(\beta n)} - \frac{e^{-x}}{n} \right) + (1-\beta) \left( 1 - e^{-x} \frac{e^{\lambda a/\beta}}{n} \right) \right)^n \lambda e^{-\lambda a} da. \end{aligned}$$

The first term in Eq. (2) clearly tends to zero as  $n \rightarrow \infty$ . The second term is bounded by

$$\int_0^{v_n} \lambda e^{-\lambda a} da = 1 - e^{-\lambda v_n} = \lambda v_n + o\left(\frac{1}{n}\right)$$

and hence the second term tends to zero also. Now

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_{v_n}^u (\beta(e^{-y/(\beta n)} - \frac{e^{-x}}{n}) + (1-\beta)(1 - \frac{e^{-x}}{n} e^{\lambda a/\beta})^n \lambda e^{-\lambda a} da \\
&= \int_0^\infty \lim_{n \rightarrow \infty} (1 - \frac{y}{n} - \frac{e^{-x}}{n} (\beta + (1-\beta)e^{\lambda a/\beta}))^n \lambda e^{-\lambda a} da \\
&= \int_0^\infty e^{-y} \exp(-e^{-x}(\beta + (1-\beta)e^{\lambda a/\beta})) \lambda e^{-\lambda a} da \\
&= e^{-y} G_\beta(x). \text{ This completes the proof.}
\end{aligned}$$

If we let  $x \rightarrow \infty$  and then let  $n \rightarrow \infty$  we see that  $\lim_{n \rightarrow \infty} P[W_n > \frac{y}{\lambda n}] = e^{-y}$  and because  $\lim_{n \rightarrow \infty} P[M_n < \frac{\beta(x + \ln(n))}{\lambda}] = G_\beta(x)$ ,  $W_n$  and  $M_n$  are asymptotically independent. If  $\hat{W}_n$  represents the minimum of  $n$  i.i.d. exponential random variables with density  $f(x) = \lambda e^{-\lambda x}$ , then  $P[\hat{W}_n > \frac{y}{\lambda n}] = e^{-y}$  for each  $n$ . So asymptotically,  $W_n$  behaves similar to  $\hat{W}_n$ . On the other hand, if  $\hat{M}_n$  is the maximum of  $n$  i.i.d. exponential variables  $P[\hat{M}_n < \frac{x + \ln n}{\lambda}] \rightarrow \exp(-e^{-x})$  as  $n \rightarrow \infty$ . So  $M_n$  does not behave at all similar to  $\hat{M}_n$ , and in fact the norming constants are different.

Let  $R_n = M_n - W_n$  and  $T_n = (M_n + W_n)/2$ .  $R_n$  is called the range and  $T_n$  is the midrange.

**Theorem 2.2.** For the EARMA (1,1) process with  $\rho = 1$  and  $1 > \beta > 0$   
 $\lim_{n \rightarrow \infty} P[R_n < \frac{\beta(x + \ln(n))}{\lambda}] = G_\beta(x)$ , (3) and  $\lim_{n \rightarrow \infty} P[T_n < \frac{\beta}{2} \frac{(x + \ln(n))}{\lambda}] = G_\beta(x)$ . (4)

**Proof.** Because  $R_n = M_n - W_n$  and  $2 T_n = M_n + W_n$  and for every  $\delta > 0$

$$\lim_{n \rightarrow \infty} P(W_n > \delta) = 0$$

direct application of Lemma 2.2.1, Galambos (Ref. 3) yields Eqs. (3) and (4).

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